q-Binomial Coefficients

Allan Silberger

Cleveland State University

October 2009

In the first part of this lecture, for comparison purposes, we recall the characterizing properties of ordinary binomial coefficients both in the binomial theorem and in "counting."

The invention of q-binomial coefficients is attributed to Euler, but Gauss made the most famous use of them in determining the "sign of the Gauss sum" before 1810. We shall introduce q-binomial coefficients as a polynomial generalization of ordinary binomial coefficients. They have their own Pascal's triangle, occur as coefficients for a "q-binomial theorem" and also play a role in counting. To describe their counting role we shall briefly discuss partitions and some related generating functions.

There are wonderful articles on binomial and q-binomial coefficients on the web. I also recommend Dr. Soprunov's lectures on Combinatorics for an elegant and deeper discussion at least of ordinary binomial coefficients.

Ideas Associated With the Usual Binomial Coefficients

What do we mean when we say, “I know what a binomial coefficient is.”?

We answer this question in some detail, because, in our second half, we want to show that q-binomial coefficients have many of the same properties.

Factorials

$$
\begin{aligned}
0! &= 1 \\
1! &= 1 \\
2! &= 2 \cdot 1 = 2 \\
3! &= 3 \cdot 2 \cdot 1 = 6 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\
\vdots &= \vdots \\
n! &= n \cdot (n-1)! = n \cdot n - 1 \cdot n - 2 \cdots 3 \cdot 2 \cdot 1
\end{aligned}
$$

$$5! = 120 \quad 6! = 720 \quad 7! = 5040 \quad 10! = 3628800 \quad 12! = 479001600$$

$$15! = 1307674368000$$

Binomial Coefficients: The Definition

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

Examples:

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \quad \binom{4}{1} = 4 \quad \binom{4}{2} = 6 \quad \binom{4}{3} = 4 \quad \binom{4}{4} = 1$$

$$\binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

Pascal's Triangle

The n-th row of Pascal's triangle lists in order the coefficients of the expansion of $(x+y)^n$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Observations

- $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$
- Formula for Pascal's Triangle:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Binomial Theorem:

$$(x+y)^0 = 1 \qquad (x+y)^1 = 1 \cdot x + 1 \cdot y \qquad (x+y)^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$

$$(x+y)^3 = 1 \cdot x^3 + 3 \cdot x^2 y + 3 \cdot xy^2 + 1 \cdot y^3$$

$$(x+y)^4 = 1 \cdot x^4 + 4 \cdot x^3 y + 6 \cdot x^2 y^2 + 4 \cdot xy^3 + 1 \cdot y^4$$

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

Counting Interpretation of Factorials and Binomial Coefficients

- $n!$ counts the number of orderings of a set of size n.
- $\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$ counts the number of **ordered choices** of k elements from a set of size n.
- $\binom{n}{k} = \frac{(n!/(n-k)!)}{k!}$ gives the number of k-membered subsets which can be chosen from a set of size n. We divide the number of ordered choices by the number of orderings for any set consisting of k elements.

An Example

$$\{a, b, c, d, e\}$$

is a set of size five. The three member subsets, listed in alphabetical order, are:

$$\{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}$$

$$\{a, d, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}$$

Each three member set has 6 different orderings. For example:

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba$$

Thus, $\binom{5}{3} = 10$ and $\frac{5!}{(5-3)!} = 10 \cdot 3! = 10 \cdot 6 = 60$.

Pascal's Triangle by Counting

The formula for Pascal's triangle is: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Justification: We want to count the number of k member subsets of a set of size n.

Fix one element. Then $n-1$ elements are left.

$\binom{n-1}{k-1}$ counts the number of k member subsets of the n elements which include the fixed element and $\binom{n-1}{k}$ gives the number of k member subsets which don't include the fixed element.

The sum of these two numbers is the number of k member subsets of a set containing n elements.
The formula for Pascal's triangle implies, via induction, that $\binom{n}{k}$ is an integer for all n, k. If $n < k$ or $k < 0$, then $\binom{n}{k} = 0$.

Binomial Theorem by a Counting Argument

- Counting:

  Expanding we have

  $$(x+y)^n = (x+y)(x+y)\cdots(x+y) \qquad (n\,\text{parentheses}).$$

  Pick x out of k parentheses and y out of $n-k$ parentheses. This can be done in $\binom{n}{k}$ ways. Thus $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$. This means that

  $$(x+y)^n = \binom{n}{0}x^0y^n + \cdots + \binom{n}{k}x^ky^{n-k} + \cdots + \binom{n}{n}x^ny^0.$$

Binomial Theorem by Pascal's Triangle

- This is simply a proof by induction:

$$(x+y)^n = (x+y)^{n-1}y + x(x+y)^{n-1}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} y + x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-(k+1)}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k} + \sum_{k=1}^{n} \binom{n-1}{k-1} x^k y^{n-k}$$

$$= \binom{n-1}{0} x^0 y^n + \sum_{k=1}^{n-1} \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) x^k y^{n-k} + \binom{n-1}{n-1} x^n y^0$$

$=\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, since $\binom{n-1}{0} = \binom{n}{0} = \binom{n}{n} = \binom{n-1}{n-1} = 1$.

q-Binomial Coefficients

How dare we call them binomial coefficients?

We try to convince you that q-binomial coefficients really are a "polynomial generalization" of the usual binomial coefficients.

$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$

Examples:

$$
\begin{aligned}
x - 1 &= x - 1 \\
x^2 - 1 &= (x-1)(x+1) \\
x^3 - 1 &= (x-1)(x^2+x+1) \\
x^4 - 1 &= (x-1)(x^3+x^2+x+1) = (x-1)(x+1)(x^2+1) \\
x^5 - 1 &= (x-1)(x^4+x^3+x^2+x+1) \\
x^6 - 1 &= (x-1)(x^5+x^4+x^3+x^2+x+1) \\
&= (x-1)(x+1)(x^2+x+1)(x^2-x+1)
\end{aligned}
$$

The q-Factorial Function

As usual we define factorials recursively:

$$
\begin{aligned}
(0)_q! &:= 1 \\
(1)_q! &:= q - 1 \\
(2)_q! &:= (q^2 - 1)(q - 1) \\
(3)_q! &:= (q^3 - 1)(q^2 - 1)(q - 1) \\
\vdots &:= \vdots \\
(n)_q! &:= (q^n - 1) \cdot (n - 1)_q! = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)
\end{aligned}
$$

We define

$$\binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!}$$
$$= \frac{(q^n-1)\cdots(q^{n-k+1}-1)}{(q-1)\cdots(q^k-1)}$$
$$= \frac{(q^{n-1}+\cdots+q+1)\cdots(q^{n-k}+\cdots+q+1)}{1\cdot(q+1)\cdots(q^{k-1}+\cdots+q+1)}$$

We'll see later that $\binom{n}{k}_q$ is a polynomial in the variable q. The first line shows that $\binom{n}{k}_q = \binom{n}{n-k}_q$. The last line of "equals" shows that, at $q=1$, $\binom{n}{k}_q = \binom{n}{k}$.

Examples

- $\binom{n}{0}_q = 1$ for all $n \geq 0$.
- $\binom{n}{1}_q = \frac{q^n-1}{q-1} = q^{n-1} + \cdots + q + 1$ for all $n \geq 1$.
- If n is even, then

$$\binom{n}{2}_q = \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1))} = \frac{q^n-1}{q^2-1}\frac{q^{n-1}-1}{q-1}$$
$$= (q^{n-2}+q^{n-4}+\cdots+q^2+1)(q^{n-2}+q^{n-3}+\cdots+q+1)$$

If n is odd, then $n-1$ is even and

$$\binom{n}{2}_q = (q^{n-1}+q^{n-2}+\cdots+q+1)(q^{n-3}+q^{n-5}+\cdots+q^2+1).$$

The Formula for Pascal's Triangle

Theorem

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q .$$

The second equivalent version of the formula follows from the first version and the symmetry:

$$\begin{aligned}\binom{n}{k}_q &= \binom{n}{n-k}_q \\ &= q^{n-k}\binom{n-1}{n-k}_q + \binom{n-1}{n-k-1}_q \\ &= q^{n-k}\binom{n-1}{k-1}_q + \binom{n-1}{k}_q\end{aligned}$$

Direct Justification for Pascal's Triangle Formula

$$
\begin{aligned}
\binom{n}{k}_q &= \frac{(n)_q!}{(k)_q(n-k)_q} \\
&= \frac{q^n - 1}{q^{n-k} - 1} \cdot \frac{(n-1)_q!}{(k)_q!(n-1-k)_q!} \\
&= \frac{q^n - q^k + q^k - 1}{q^{n-k} - 1} \binom{n-1}{k}_q \\
&= q^k \binom{n-1}{k}_q + \frac{q^k - 1}{q^{n-k} - 1} \frac{(n-1)_q!}{(k)_q!(n-1-k)_q!} \\
&= q^k \binom{n-1}{k}_q + \frac{(n-1)_q!}{(k-1)_q!(n-k)_q!} \\
&= q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q
\end{aligned}
$$

The q-Binomial Theorem

Theorem

$$(x+y)(x+qy)(x+q^2y)\cdots(x+q^{n-1}y)=\sum_{k=0}^{n} q^{k(k-1)/2}\binom{n}{k}_q x^{n-k}y^k.$$

For the proof imitate the inductive proof we gave for the usual binomial theorem, substituting the formula for Pascal's triangle in the q-binomial case.

Counting interpretations:

- Let $q := p^s$ be a prime power. Then there is a finite field $\mathbb{F}_q$ containing q elements and a vector space $\mathbb{F}_q^n$ which contains q^n elements.

  FACT: $\binom{n}{k}_q$ is the number of distinct k-dimensional subspaces of $\mathbb{F}_q^n$.

  This "FACT" is, in fact, useful for the following reason. In coding theory we study "error correcting codes." A code is generally a k-dimensional subspace of some $\mathbb{F}_q^n$. Suppose we have a random "word" in $\mathbb{F}_q^n$. Then $1/\binom{n}{k}_q$ is the the probability that a random word will be regarded as a code word.

Partitions; Examples of Partitions

A *partition of* $n \geq 0$ is a multiset consisting of positive numbers whose sum is n.

Examples:
The only partition of 0 is $\emptyset$ (the "empty set").
The only partition of 1 is the set $\{1\}$.
The partitions of 2 are 2 and 1,1.
The partitions of 3: 3; 2,1; 1,1,1.
The partitions of 4: 4; 3,1; 2,2; 2,1,1; and 1,1,1,1.
The partitions of 5: 5; 4,1; 3,2; 3,1,1; 2,2,1; 2,1,1,1; and 1,1,1,1,1.
The partitions of 10: 10; 91; 82; 811; 73; 721; 7111; 64; 631; 622; 6211; 61111; 55; 541; 532; 5311; 5221; 52111; 511111; 442; 4411; 433; 4321; 43111; 4222; 42211; 421111; 4111111; 3331; 3322; 33211; 331111; 32221; 322111; 3211111; 31111111; 22222; 222211; 2221111; 22111111; 211111111; 1111111111

The *canonical form of a partition* $\lambda := \ell_1 \geq \ell_2 \geq \cdots \geq \ell_r > 0$. In the examples we listed partitions in their canonical forms.

Another notation: $1^{\alpha_1} 2^{\alpha_2} 3_3^{\alpha} \cdots$.
Example: $54221 = 1^1 2^2 3^0 4^1 5^1$.

The *weight* of partition λ is $\ell_1 + \ell_2 + \cdots + \ell_r$.
Write $|\lambda|$ for the weight of λ.

The *rank* of a partition is the number of parts.
Write $r(\lambda)$ for the rank of λ.

Example: Let $\lambda = 54221$. Then $|\lambda| = 14$ and $r(\lambda) = 5$.

Write $p(n)$ for the number of partitions of weight n.
Examples: $p(0) = 1$, $p(1) = 1$, $p(3) = 3$, $p(5) = 7$ and $p(10) = 42$.

Review of Partitions

Examples (expressed in inverse lexicographic order):

- The unique partition of 0 has no nonzero parts.
- 1 is the unique partition of 1.
- 2 and 1,1 are the partitions of two.
- 3; 2,1; 1,1,1 are the partitions of 3.
- 4; 3,1; 2,2; 2,1,1; 1,1,1,1 are the partitions of 4.

The *parts* are the positive numbers occurring in the partition. They can be listed in any order. The partition of 0 has 0 parts. The *weight* of 3,2,1 is 6 because 3+2+1=6.

The *rank* of 4,3,3,2,2,1 is 6, because there are 6 parts. The weight is 15.

Generating function for $p(n)$

$P(x) := \sum_{k=0}^{\infty} p(k)x^k$ is the generating function of the sequence $\{p(n)\}_{n=0}^{\infty}$.

$$\begin{aligned}
\sum_{k=0}^{\infty} p(k)x^k &= \prod_{n=1}^{\infty} \frac{1}{1-x^n} \\
&= \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} \\
&= \prod_{n=1}^{\infty} (1+x^n+x^{2n}+x^{3n}+\cdots) \\
&= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots \\
&= 1+x+2x^2+3x^3+5x^4+7x^5+\cdots+42x^{10}+\cdots+p(n)x^n+\cdots
\end{aligned}$$

A Theorem of Euler

Theorem

Let $q(n)$ [$r(n)$] denote the number of partitions of n such that each part is an odd number [such that the canonical form satisfies $\ell_1 > \ell_2 > \cdots > \ell_r > 0$.] Then $q(n) = r(n)$ for all $n \geq 0$.

Proof.

Euler's proof is a wonderfully easy demonstration of the power of generating functions: Let $Q(x) := \sum_{k=0}^{\infty} q(k)x^k$ and $R(x) := \sum_{k=0}^{\infty} r(k)x^k$ be the respective generating functions. Then $Q(x) = R(x)$, since

$$Q(x) = \prod_{k=0}^{\infty} \frac{1}{1 - x^{2k+1}} = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots},$$

$$R(x) = \prod_{k=1}^{\infty}(1 + x^k) = (1+x)(1+x^2)(1+x^3)\cdots .$$

Lemma

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

Proof.

$$\begin{aligned}(1+x)(1+x^2)(1+x^3)\cdots &= \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}\\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}\end{aligned}$$

□

Theorem

The polynomial $\binom{n}{k}_q$ is a generating function for the number of partitions of $j \geq 0$ with no more than $n-k$ (nonzero) "parts" such that each part is no more than k, or, equivalently, no more than k parts such that each part is no more than $n-k$.

EXAMPLE:

$$\binom{n}{1}_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

If every part is no more than 1 and there are no more than $n-1$ parts, then there is 1 partition (of 0) with 0 parts, 1 partition (of 1) with 1 part, 1 partition of 2 with 2 parts, etc.

If every part is no more than $n-1$ and there is at most one nonzero part, then we have a unique partion of 0 again, 1 partition of 1, 1 partition of 2, 1 partition of 3, etc. up to 1 partition of $n-1$.

Another, more explicit example; an asymptotic formula

$$\binom{5}{2}_q = \frac{(q^5-1)(q^4-1)}{(q-1)(q^2-1)}$$
$$= (q^4+q^3+q^2+q+1)(q^2+1)$$
$$= q^6+q^5+2q^4+2q^3+2q^2+q+1$$
$$\sim 222; 221; 22, 211; 21, 111; 2, 11; 1; \emptyset$$
$$\sim 33; 32; 22, 31; 21, 3; 2, 11; 1; \emptyset$$

In the limit,

$$\lim_{n\to\infty}\binom{2n}{n}_q = P(q).$$

Proof.

The proof is by induction and is based on the formula for Pascal's Triangle in the q-binomial case:

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q .$$

By the induction hypothesis the term $\binom{n-1}{k-1}_q = 1 + a_1 q + \cdots + a_j q^j + \cdots$, where a_j is the number of partitions of weight j such that each part is $\leq k-1$ and there are no more than $n-k = (n-1)-(k-1)$ parts. Similarly, the term $q^k \binom{n-1}{k}_q = q^k + b_{k+1} q^{k+1} + \cdots + b_j q^j + \cdots$ is the number of partitions of weight j such that each part is $\leq k$, at least one part is k, and there are $\leq n-k$ parts. Adding these two polynomials together we get exactly the generating function such that the coefficients $a_j + b_j$ represent exactly the number of partitions of j such that each part is $\leq k$ and there are $\leq n-k$ parts. $\square$